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UMBRAL DEFORMATIONS TO DISCRETE SPACETIME

We probably live on some sort of **discrete spacetime lattice**, at scales of order $l_{Planck} \equiv \sqrt{\hbar G_N/c^3} \sim 1.6 \cdot 10^{-35} m$, the corresponding time for which is $l_{Planck}/c \sim 5.4 \cdot 10^{-44} sec$, and the mass $m_{Planck} = \frac{\hbar}{c \; l_{Planck}} \sim 2.2 \cdot 10^{-5} g$.

Could we tell? ▶ Likely:

We can discretize all of continuum physics systematically through the **Umbral Deformation**.

REVIEW OF ARGUMENTS FOR A
FUNDAMENTAL MINIMUM MEASURABLE LENGTH

(In geometrical Planck units: $\hbar$, c, $m_{Planck} \equiv \sqrt{\hbar c/G_N}$ chosen to be 1).

In a system or process characterized by energy E, no lengths **smaller** than L can be accessed or measured, where L is the **larger** of either the Schwarzschild horizon radius of the system, $R_{Sch} \sim E$, or, for energies smaller than m_{Planck}, the Compton wavelength of the aggregate process, $\lambda \sim 1/E$.

Since the minimum of

$$\boxed{\text{max}\ (E, \tfrac{1}{E})}$$

is at the Planck mass, $E \sim 1$, $\rightsquigarrow$ the **smallest measurable distance** L amounts to l_{Planck}.

$\circledcirc$ For the earth, $R_{Sch} \sim 0.9cm$; for me, $R_{Sch} \sim 5 \cdot 10^{-26} m = 0.5 \cdot 10^{-10} f$.

Thus, continuum laws in nature are expected to be deformed, in principle, by modifications of $O(l_{Planck})$.

• Even as something like a fundamental spacetime lattice with spacing $a = O(l_{Planck})$ is likely to underlie nature, continuous symmetries (such as Galilei or Lorentz invariance) can actually **survive unbroken such a deformation into discreteness**, in a nonlocal **umbral realization**.

▶ **Umbral calculus**, pioneered by Rota & associates in combinatorics contexts, specifies how functions of discrete variables representing observables get to "shadow" their continuum limit ($a \to 0$) systematically; and effectively preserve Leibniz's chain rule **and the Lie Algebras** of the difference operators which shadow (deform) the standard differential operators of continuum physics. (Reviewed by Levi & Winternitz.)

⋔ Nevertheless, while the continuous symmetry algebras of umbrally deformed systems may remain identical to their continuum limit, the **functions of observables themselves are modified**, usually drastically so. Often, the continuum differential equations of physics are discretized; then, solved to yield umbral deformations of the continuum solutions. Complication may be bypassed by umbrally **deforming the continuum solutions directly**.

OVERVIEW OF THE UMBRAL CORRESPONDENCE

Consider discrete time, $t = 0, a, 2a, ..., na, ...$ Without loss of generality, consider first the Δ_+ discretization (umbral deformation) of ∂_t,

$$\Delta x(t) \equiv \frac{x(t+a) - x(t)}{a} ,$$

$\rightsquigarrow$ elementary oscillations, $\ddot{x}(t) = -x(t)$:

$$\Delta^2 x(t) = \frac{x(t+2a) - 2x(t+a) + x(t)}{a^2} = -x(t) .$$

Fourier-component Ansatz $x(t) \propto r^t$, $\rightsquigarrow$ $(1 \pm a)^{t/a}$.
Alternatively, in the umbral framework: associative chains of operators, generalizing ordinary continuum functions, by ultimately acting on a translationally-invariant "Fock vacuum", 1.

For the standard Lagrange-Boole shift generator,

$T \equiv e^{a\partial_t}$, so that $Tf(t) \cdot 1 = f(t+a)T \cdot 1 = f(t+a)$,

the umbral deformation is

$$\partial_t \mapsto \Delta \equiv \frac{T-1}{a} ; \qquad t \mapsto tT^{-1},$$

$$t^n \quad \mapsto \quad (tT^{-1})^n = t(t-a)(t-2a)...(t-(n-1)a)T^{-n} \equiv [t]^n T^{-n},$$

("basic polynomials"), $\rightsquigarrow$ $[t]^0 = 1$, and, for $n > 0$, $[0]^n = 0$.

$\looparrowright$ A linear combination of monomials (power series representation of a function) will transform umbrally to the **same linear combination of basic polynomials with the same series coefficients**,

$f(t) \quad \mapsto \quad f(tT^{-1})$.

All observables $F(t)$ in the discretized world are thus such deformation maps of the continuum observables. (Eliminate translation operators at the very end, through operating on 1, so that $f(tT^{-1}) \cdot 1 \equiv F(t)$.)

The umbral deformation relies on the respective umbral entities obeying operator combinatorics **identical to their continuum limit** ($a \to 0$), by virtue of obeying the **same Heisenberg commutation relation**,

$$\boxed{[\partial_t, t] \; = \; \mathbb{1} = \; [\Delta, tT^{-1}]}.$$

(Unitary equivalences of the unitary irrep of the Heisenberg-Weyl group, provided for by the Stone-von Neumann theorem. ∞-dim.)

By shift invariance, $T\Delta T^{-1} = \Delta$,

$$[\partial_t, t^n] = nt^{n-1} \qquad \mapsto \qquad [\Delta, [t]^n T^{-n}] = n[t]^{n-1}\ T^{1-n},$$

so, ultimately, $\quad \Delta[t]^n = n[t]^{n-1}$.

$$[t]^n T^{-n}[t]^m T^{-m} \equiv [t]^n \circledast [t]^m T^{-n-m} = [t]^{n+m} T^{-(n+m)},$$

Implicit definition of the product through dotting on 1, $[t]^n \circledast [t]^m \equiv [t]^{n+m}$.

For commutators of associative operators, the umbrally deformed Leibniz rule holds,

$$[\Delta, f(tT^{-1})g(tT^{-1})] = [\Delta, f(tT^{-1})]g(tT^{-1}) + f(tT^{-1})[\Delta, g(tT^{-1})]\ ,$$

ultimately to be dotted onto 1.

⊛ The basic polynomials $[t]^n$ are just scaled falling factorials $a^n (t)_n$,

$$[t]^n = a^n \frac{(t/a)!}{(t/a-n)!} \ ,$$

so $[-t]^n = (-)^n [t + a(n-1)]^n$. Also, $[an]^n = a^n n!$; for $0 \le m \le n$, $[t]^m [t - am]^{n-m} = [t]^n$; for integers $0 \le m < n$, $[am]^n = 0$; $\Delta^m [t]^n = [an]^m [t]^{n-m} / a^m$.

The standard **umbral exponential**,

$$E(\lambda t, a) \equiv e^{\lambda [t]} \equiv e^{\lambda t T^{-1}} \cdot 1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [t]^n = \sum_{n=0}^{\infty} (\lambda a)^n \binom{t/a}{n} = (1 + \lambda a)^{t/a},$$

the **compound interest formula**, with the proper continuum limit $(a \to 0)$. Since $\Delta \cdot 1 = 0$,

$$\Delta e^{\lambda [t]} = \lambda \ e^{\lambda [t]}.$$

Could have solved this equation directly to produce the above $E(\lambda t, a)$.

⊛ The umbral exponential E **happens to be an ordinary exponential**,

$$e^{\lambda [t]} = e^{\frac{\ln(1+\lambda a)}{a} t}.$$

The umbral exponential serves as the generating function of the umbral basic polynomials,

$$\frac{\partial^n}{\partial\lambda^n}(1+\lambda a)^{t/a}\Big|_{\lambda=0} = [t]^n .$$

Conversely, this may be reversed, by first solving directly for the umbral eigenfunction of Δ, and effectively defining the umbral basic polynomials through these parametric derivatives, in situations where these might be more involved.

By linearity, the umbral deformation of a power series representation of a function formally evaluates to

$$f(t) \quad \mapsto \quad F(t) \equiv f(tT^{-1}) \cdot 1 = f\left(\frac{\partial}{\partial\lambda}\right)\ (1+\lambda a)^{t/a}\Big|_{\lambda=0} .$$

Can use **Fourier representation** instead. $\Rrightarrow$ Same argument, now on linear combinations of exponentials $\leadsto$

$$\boxed{F(t) = \int_{-\infty}^{\infty} d\tau f(\tau) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i\tau k}(1+ika)^{t/a} = \left(1 + a\frac{\partial}{\partial\tau}\right)^{t/a}\ f(\tau)\ \Big|_{\tau=0}} .$$

The rightmost equation follows by converting k into ∂_τ derivatives and integrating by parts away from the resulting delta function. (Connects to above by the Fourier identity $f(\partial_x)g(x)|_{x=0} = g(\partial_x)f(x)|_{x=0}$.)

▶ Check this umbral transform functional yields

$$\partial_t f \;\mapsto\; \Delta F \;;$$

$$\delta(t) \;\mapsto\; \frac{\sin(\frac{\pi}{2}(1+t/a))}{(\pi(a+t))} \;;$$

$$f = \frac{1}{(1-t)} \;\mapsto\; F = e^{1/a} a^{t/a} \Gamma(t/a+1, 1/a) \;,$$

an incomplete Gamma function, etc. More direct, in general.

E.g., for trigonometric functions,

$$\sin[t] \equiv \frac{e^{i[t]} - e^{-i[t]}}{2i} \;, \qquad \cos[t] \equiv \frac{e^{i[t]} + e^{-i[t]}}{2} \;,$$

$\leadsto$

$$\Delta \sin[t] = \cos[t] \;, \qquad \Delta \cos[t] = -\sin[t] \;.$$

Thus, the umbral deformation of phase-space rotations,

$$\dot{x} = p, \quad \dot{p} = -x \qquad \mapsto \qquad \Delta x(t) = p(t), \quad \Delta p(t) = -x(t),$$

readily yields, by directly deforming continuum solutions, oscillatory solutions,

$$x(t) = x(0)\cos[t] + p(0)\sin[t], \qquad p(t) = p(0)\cos[t] - x(0)\sin[t].$$

Since

$$(1+ia) = \sqrt{1+a^2}\; e^{i\,\mathrm{arctan}(a)},$$

↬ discrete phase-space **spirals**,

$$x(t) = (1+a^2)^{\frac{t}{2a}}\,\Big(x(0)\cos(\omega t) + p(0)\sin(\omega t)\Big),$$

$$p(t) = (1+a^2)^{\frac{t}{2a}}\,\Big(p(0)\cos(\omega t) - x(0)\sin(\omega t)\Big).$$

↬ **Frequency decreased** from the continuum value 1 to

$$\boxed{\omega = \mathrm{arctan}(a)/a \leq 1}\ ,$$

effectively the inverse of the cardinal tangent function.

$\circledcirc$ For $\theta \equiv \text{arctan}(a)$, the spacing of the zeros, period, etc, are **scaled up** by a factor of

$$\text{tanc}(\theta) \equiv \frac{\tan(\theta)}{\theta} \geq 1 \ .$$

▶ The umbrally conserved quantity is,

$$2\mathcal{E} = x(t) \divideontimes x(t) + p(t) \divideontimes p(t) = x(0)^2 + p(0)^2 = (1+a^2)^{\frac{-t}{a}}(x(t)^2 + p(t)^2),$$

$(\Delta\mathcal{E} = 0)$, with the proper energy as the continuum limit.

MORE SYMMETRIC CASES

The previous Δ_+ is not time-reversal odd, and thus its square is not time-reversal invariant—whence the awkward outspiraling of the solutions seen.

Can fix this by choosing the time-reversal-odd umbral deformation,

$$\boxed{\partial_t \qquad \mapsto \qquad \Delta_s \equiv \frac{T-T^{-1}}{2a}} \quad .$$

The eigenfunctions of $\Delta_s E_s = \lambda E_s$ are now two,

$$E_{s\pm} = \left(\lambda a \pm \sqrt{1+(\lambda a)^2}\right)^{t/a} \quad ;$$

one, E_{s+}, going to the exponential in the continuum limit; but the other, E_{s-} (**nonumbral**), simply oscillating to zero—an oscillation of infinite frequency.

Since $[\Delta_s, t\ 2/(T+T^{-1})] = 1\!\!1$, the basic polynomials, $[t]^n_s = (t\ \ 2/(T+T^{-1}))^n \cdot 1$, would be harder to evaluate; instead, evaluated from $\Delta_s [t]^n_s = n[t]^{n-1}_s$,

$$[t]^n_s = t \prod_{k=1}^{n-1} \Big(t + a(n-2k)\Big).$$

$\looparrowright$ Shortcut: they may alternatively be generated more directly from the generating function E_{s+},

$$[t]_s^n = \frac{\partial^n}{\partial\lambda^n}\left(\lambda a + \sqrt{1+(\lambda a)^2}\right)^{t/a}\Big|_{\lambda=0}.$$

E.g., check that $\quad [t]_s^3 = (t+a)t(t-a), \quad$ etc.

▶ Rather than using umbral deformations of power series representations for functions, however, one may again instead infer umbral transforms of **Fourier representations**,

$$f(t) \qquad \mapsto \qquad F_s(t) = \int_{-\infty}^{\infty} d\tau\, f(\tau) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i\tau k}\left(ika + \sqrt{1-(ka)^2}\right)^{t/a},$$

to evaluate umbral deformations for general observables, as well as non-umbral ones relying on E_{s-}, with a minus sign in the kernel of this deforming functional.

Thus, now there are **four solutions** to

$$(\Delta_s^2 + 1)x(t) = 0\ , \qquad \rightsquigarrow$$

$$x(t) = \left(\pm ia \pm \sqrt{1-a^2}\right)^{t/a}\ .$$

$\rightsquigarrow$ The discrete-time solution set

$$x(t) = (-)^{Nt/a}\Big(x(0)\cos(\omega t) + p(0)\sin(\omega t)\Big),$$

$$p(t) = (-)^{Nt/a}\Big(p(0)\cos(\omega t) - x(0)\sin(\omega t)\Big),$$

maps onto itself under time-reversal, for integer parameter $N = 0, 1$. ($\Delta_s x(0) = (-)^N p(0)$.) Eqn only connects even points on the time lattice among themselves, and odd ones among themselves. $\rightsquigarrow$ all even points on the time lattice behave the same for even or odd parameter N. (However, for the $N = 1$ solutions, the odd time points hop out of phase by π, reflection with respect to the origin in phase space, as they are not dynamically linked to the even points—a phenomenon familiar in lattice gauge theory.)

▲ Actually, if $f(t)$ is a solution, $g(t)f(t)$ will also be a solution for arbitrary periodic $g(t+2a) = g(t)$. Thus, even though $(-)^{t/a}$ is one such possible $g(t)$, there are even **more** solutions with arbitrarily mismatched moduli (phase-space radii) and phases between the odd and even sublattices—only their frequencies of rotation need be the same. The solution set is 4-dimensional.

For $N = 0$, the frequency is **increased** over its continuum limit:

$$\boxed{\omega = \text{arcsin}(a)/a \geq 1}\ .$$

(For $N = 1$, the arcsine effectively advances by π and the frequency has an additional component of π/a. Thus, these nonumbral solutions collapse to 0 in the continuum limit.)

The conserved energy is more conventional,

$$2\mathcal{E}_s = x(t)^2 + p(t)^2.$$

This time-reversal-odd difference operator is the one to be considered in wave propagation, to avoid presumably unphysical exponential amplitude modulations, growths or dwindlings, peculiar to the asymmetric derivative seen.

WAVE PROPAGATION

▲ Simple plane waves in a positive or negative direction x would obey an equation of the type

$$(\Delta_x^2 - \Delta_t^2)\ F = 0,$$

with the symmetric difference operators Δ_s on a time lattice with spacing a, and an x-lattice of spacing b, respectively, not necessarily such that $b = a$ in some spacetime regions.

For generic frequency, wavenumber and velocity, the basic right-moving waves $e^{i(\omega t - kx)}$ have phase velocity

$$\boxed{v_s(\omega, k) = \frac{\omega}{k}\ \frac{a \arcsin(b)}{b \arcsin(a)} \quad ,}$$

that is to say, the effective index of refraction in the discrete medium is $(b \arcsin(a))/(a \arcsin(b))$, so modified from 1 by $O(l_P)$.

• Small inhomogeneities of a and b in the fabric of spacetime over large regions could yield interesting **frequency shifts in the index of refraction**, and thus, e.g., whistler waves over cosmic distances. It might be worth investigating application of the umbral deformation functional on such waves, to access long range effects of microscopic qualifications of the type considered.

A more technically challenging application of the umbral tranforms proposed might attain significance on **nonlinear solitonic phenomena**, such as, e.g., the one-soliton solution of the continuum Sine-Gordon equation,

$$\left(\partial_x^2 - \partial_t^2\right) f(x,t) = \sin(f).$$

The corresponding umbral deformation of the equation itself would now also involve a deformed potential $\sin\left(f(x\ \frac{2}{T_x+T_x^{-1}}, t\ \frac{2}{T_t+T_t^{-1}})\right) \cdot 1$ on the right-hand side, for the Δ_s deformation — and $\sin\left(f(xT_x^{-1}, tT_t^{-1})\right) \cdot 1$ for the Δ_+ deformation.

$\rightsquigarrow$ Rather than solving difficult nonlinear difference equations, may instead infer the umbral transform of, e.g., the continuum one-soliton **solution**,

$$F_s = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk}{\pi^2} \arctan\left(m e^{\frac{\chi - v\tau}{\sqrt{1-v^2}}}\right)\ e^{-i\chi p - i\tau k}$$

$$\left(ipb + \sqrt{1-(pb)^2}\right)^{x/b} \left(ika + \sqrt{1-(ka)^2}\right)^{t/a}.$$

For the Δ_+ deformation, instead,

$$F_+ = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk}{\pi^2} \arctan\left(m e^{\frac{\chi - v\tau}{\sqrt{1-v^2}}} \right) \ e^{-i\chi p - i\tau k} \Big(ipb + 1\Big)^{x/b} \Big(ika + 1\Big)^{t/a}.$$

Closed form evaluation complicated, but maybe plotted numerically ... Could yield qualitative asymptotic insights on the $O(l_{Planck})$ modifications of such "umbral solitons"?

Likewise, the analog integrals with the **continuum KdV soliton** $f(x,t) = \frac{v}{2}\ \mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt)\right)$ as input,

$$F_s = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk\ v}{8\pi^2} \mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(\chi - v\tau)\right) \ e^{-i\chi p - i\tau k}$$

$$\Big(ipb + \sqrt{1-(pb)^2}\Big)^{x/b} \Big(ika + \sqrt{1-(ka)^2}\Big)^{t/a};$$

and, for the Δ_+ deformation,

$$F_+ = \int_{-\infty}^{\infty} \frac{d\chi d\tau dp dk\ v}{8\pi^2} \mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(\chi - v\tau)\right) e^{-i\chi p - i\tau k} (ipb+1)^{x/b} (ika+1)^{t/a},$$

could be plotted numerically and compared to the Lax pair integrability machinery. Any ideas?